

# Multiple periodic solutions of Hamiltonian systems with prescribed energy <sup>☆</sup>

Tianqing An

*Department of Applied Mathematics, Hohai University, Nanjing 210098, PR China*

Received 28 March 2006; revised 26 December 2006

Available online 27 January 2007

---

## Abstract

Consider the periodic solutions of autonomous Hamiltonian systems  $\dot{x} = J\nabla H(x)$  on the given compact energy hypersurface  $\Sigma = H^{-1}(1)$ . If  $\Sigma$  is convex or star-shaped, there have been many remarkable contributions for existence and multiplicity of periodic solutions. It is a hard problem to discuss the multiplicity on general hypersurfaces of contact type. In this paper we prove a multiplicity result for periodic solutions on a special class of hypersurfaces of contact type more general than star-shaped ones.

© 2007 Elsevier Inc. All rights reserved.

MSC: 58E05; 58F05; 34C25

**Keywords:** Critical point; Hamiltonian system; Hypersurfaces of contact type; Periodic solution; Positive-type hypersurface

---

## 1. Introduction and main result

This paper considers the periodic solutions of autonomous Hamiltonian system

$$\dot{x} = J\nabla H(x) \tag{1.1}$$

on the given compact energy hypersurface  $\Sigma = H^{-1}(1) \subset \mathbf{R}^{2n}$ . Where  $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}^1)$  and  $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$  is the standard symplectic matrix. In 1978, P.H. Rabinowitz [18] and Weinstein [22] proved the existence of at least one periodic solution on  $\Sigma$  provided  $\Sigma$  is star-shaped (or con-

---

<sup>☆</sup> Partially supported by NNSF of China (10571085) and Science Foundation of Hohai University.  
E-mail address: [antq@hhu.edu.cn](mailto:antq@hhu.edu.cn).

vex). Since then, many remarkable results for existence and multiplicity of periodic solutions have been contributed. See for example [7–9,13,16,17] for multiplicity on convex hypersurfaces, [6,12,15] for multiplicity on star-shaped hypersurfaces and [14,20] for existence on hypersurfaces of contact type. There are also some nonexistence and almost existence results (for a family of hypersurfaces), please consult [11] and [19] respectively.

The first multiplicity result for hypersurfaces more general than star-shaped ones was given by the author in [1]. In the sequel, [2] proved some new conclusions. The purpose of this paper is to consider the multiplicity of periodic solutions of (1.1) on a class of hypersurfaces of contact type which are different from that of [1,2].

Denote the inner product and norm of  $\mathbf{R}^{2n}$  by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  respectively. A vector field  $V$  defined on  $\mathbf{R}^{2n}$  is called positive if  $\langle V(x), x \rangle > 0$  for  $x \in \mathbf{R}^{2n} \setminus \{0\}$ .

**Definition 1.1.** (See [1].) Let  $\Sigma$  be a  $C^r$  ( $r \geq 1$ ) compact connected hypersurface in  $\mathbf{R}^{2n}$ . We call  $\Sigma$  a positive-type hypersurface if there is a continuous positive vector field  $V$  transverse to  $\Sigma$ .

In [1,2] we obtained the multiplicity results on positive-type hypersurfaces under the additional assumptions that  $V$  is linear and satisfies:

$$(V_1) \quad JV = VJ,$$

$$(V_2) \quad \langle V(x), x \rangle = C_0 \langle x, x \rangle \text{ for some constant } C_0 > 0 \text{ and for } \forall x \in \mathbf{R}^{2n}.$$

The conditions  $(V_1)$  and  $(V_2)$  imply that  $\Sigma$  is of contact type, but not necessarily star-shaped. A natural new problem is to consider the case that the positive linear vector field  $V$  does not satisfy  $(V_1)$  and  $(V_2)$ .

Motivated by the work of Felmer [10], we discuss the positive-type hypersurfaces transverse to the vector fields of the following form:

$$V(x) = \begin{pmatrix} aI_n & 0 \\ 0 & bI_n \end{pmatrix} x, \quad (1.2)$$

where  $a, b$  are positive constants. It is clear that  $V$  does not satisfy  $(V_1)$  and  $(V_2)$  unless  $a = b$ . Such a class of hypersurfaces is more general than star-shaped ones and is different from the normalized positive-type hypersurfaces of [1,2], see Examples 1.7 and 1.8 below. Without loss of generality, we suppose  $a \leq b$ . It is clear that the vector field  $V$  defined by (1.2) satisfies

$$a|x| \leq |V(x)| \leq b|x|. \quad (1.3)$$

The flow  $\{\varphi_s\}$  of the vector field  $V$  is

$$\varphi_s = \begin{pmatrix} e^{as} I_n & 0 \\ 0 & e^{bs} I_n \end{pmatrix}. \quad (1.4)$$

**Proposition 1.2.** Suppose  $\Sigma \subset \mathbf{R}^{2n}$  is a positive-type hypersurface transverse to a positive vector field  $V$  defined by (1.2). Then  $\Sigma$  is of contact type and, consequently, carries a closed characteristic.

**Proof.** Let  $\bar{V} = \frac{1}{a+b}V$ , then  $\bar{V}$  is also transverse to  $\Sigma$ . The flow  $\{\bar{\varphi}_s\}$  of  $\bar{V}$  is

$$\bar{\varphi}_s = \begin{pmatrix} e^{\frac{a}{a+b}s} I_n & 0 \\ 0 & e^{\frac{b}{a+b}s} I_n \end{pmatrix}.$$

Direct computation shows

$$\langle J\bar{\varphi}_s x, \bar{\varphi}_s y \rangle = e^s \langle Jx, y \rangle.$$

Thus  $\bar{V}$  is a symplectic dilation (see [21] for definition) and  $\Sigma$  is of contact type. By [14,20],  $\Sigma$  must carry a closed characteristic.  $\square$

To consider the multiplicity of closed characteristics on a hypersurface, the key step is to construct its Hamiltonian function. By [1], if  $\Sigma$  is a  $C^r$  positive-type hypersurface, for every  $x \in \mathbf{R}^{2n} \setminus \{0\}$ , there exists a unique  $\psi(x) \in \mathbf{R}^1$  such that  $\varphi_{\psi(x)}(x) \in \Sigma$ . Define the projection  $P$  from  $\mathbf{R}^{2n} \setminus \{0\}$  to  $\Sigma$  by

$$P(x) = \varphi_{\psi(x)} x, \quad x \in \mathbf{R}^{2n} \setminus \{0\}. \quad (1.5)$$

Then  $P$  is a  $C^r$ -mapping and  $P|_S : S \rightarrow \Sigma$  is a  $C^r$ -diffeomorphism, where  $S$  is the unit sphere in  $\mathbf{R}^{2n}$ .

**Definition 1.3.** (See [1].) Define a function  $H : \mathbf{R}^{2n} \rightarrow \mathbf{R}^1$  by

$$H(x) = \begin{cases} |x|^2 / |P(x)|^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad (1.6)$$

We call it the Hamiltonian function of  $\Sigma$ .

**Lemma 1.4.** If  $\Sigma \subset \mathbf{R}^{2n}$  is of  $C^2$ , then  $H$  satisfies the following properties:

- (i)  $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}^1) \cap C^{1, \text{lip}}(\mathbf{R}^{2n}, \mathbf{R}^1)$ .
- (ii) For  $\forall x \in \mathbf{R}^{2n}$ , there hold

$$e^{2as} H(x) \leq H(\varphi_s x) \leq e^{2bs} H(x), \quad s \geq 0, \quad (1.7)$$

$$2aH(x) \leq \langle \nabla H(x), V(x) \rangle \leq 2bH(x), \quad (1.8)$$

$$\nabla H(\varphi_s x) = \varphi_s \nabla H(x), \quad (1.9)$$

$$H''(\varphi_s x) = \varphi_s H''(x) \varphi_s^{-1}, \quad x \neq 0. \quad (1.10)$$

- (iii) Let  $R = \sup_{x \in \Sigma} |x|$ ,  $r = \inf_{x \in \Sigma} |x|$ , then

$$|x|^2 / R^2 \leq H(x) \leq |x|^2 / r^2. \quad (1.11)$$

(iv) Let  $M = \sup_{x \in \Sigma} |\nabla H(x)|$ ,  $r_0 = \inf_{x \in \Sigma} \frac{\langle V(x), N(x) \rangle |x|}{|V(x)|}$ , where  $N(x)$  is the unit outward normal vector to  $\Sigma$  at  $x$ . Then

$$M \leq \frac{2b}{ar_0}. \quad (1.12)$$

**Proof.** (i) Since the mapping  $P$  of (1.5) is of  $C^2$ ,  $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}^1)$ . The conclusion  $H \in C^{1, \text{lip}}(\mathbf{R}^{2n}, \mathbf{R}^1)$  will be proved below.

(ii) For  $x = (p, q) \in \mathbf{R}^{2n} \setminus \{0\}$ ,

$$H(\varphi_s x) = H(e^{as} p, e^{bs} q) = \begin{cases} \frac{e^{2as}|p|^2 + e^{2bs}|q|^2}{|P(x)|^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then (1.7)–(1.9) come from the direct computations. By (1.9) and the fact that  $\varphi_s$  is linear for fixed  $s$ , one has

$$\langle H''(\varphi_s x) \varphi_s \xi, \eta \rangle = \langle \varphi_s H''(x) \xi, \eta \rangle, \quad \forall \xi, \eta \in \mathbf{R}^{2n}.$$

Hence, (1.10) follows.

Since  $\Sigma$  is compact, there exists a constant  $C > 0$  such that

$$\|H''(x)\| \leq C, \quad \frac{|\nabla H(x)|}{|x|} \leq C, \quad \forall x \in \Sigma.$$

Where  $\|H''(x)\|$  is the operator norm of  $H''(x): \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ .

For any  $x \in \mathbf{R}^{2n} \setminus \{0\}$ ,  $\varphi_{\psi(x)} x \in \Sigma$ . By (1.10),

$$\|H''(x)\| = \|\varphi_{\psi(x)}^{-1} H''(\varphi_{\psi(x)} x) \varphi_{\psi(x)}\| = \|H''(\varphi_{\psi(x)} x)\| \leq C.$$

So  $H \in C^{1, \text{lip}}(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}^1)$ . On the other hand, if  $\psi(x) \geq 0$ ,

$$e^{a\psi(x)} |x| \leq |\varphi_{\psi(x)} x| \leq e^{b\psi(x)} |x|.$$

Then for  $x \neq 0$ ,

$$\frac{|\nabla H(x)|}{|x|} \leq \frac{e^{a\psi(x)} |\nabla H(x)|}{e^{b\psi(x)} |x|} \leq \frac{|\nabla H(\varphi_{\psi(x)} x)|}{|\varphi_{\psi(x)} x|} = \frac{|\nabla H(P(x))|}{|P(x)|} \leq C.$$

If  $\psi(x) < 0$ , exchange the positions of  $a$  and  $b$ , the inequality still holds. Thus,

$$|\nabla H(x) - \nabla H(0)| \leq C|x - 0|, \quad \forall x \neq 0.$$

This proves  $H \in C^{1, \text{lip}}(\mathbf{R}^{2n}, \mathbf{R}^1)$ .

(iii) (1.11) comes from (1.6) immediately.

(iv) By (1.2) and (1.8),

$$\begin{aligned}
 r_0 &= \inf_{x \in \Sigma} \frac{\langle V(x), N(x) \rangle |x|}{|V(x)|} = \inf_{x \in \Sigma} \frac{\langle V(x), \nabla H(x) \rangle |x|}{|V(x)| |\nabla H(x)|} \\
 &\leq \inf_{x \in \Sigma} \frac{2b|x|}{|V(x)| |\nabla H(x)|} \leq \frac{2b}{a} \inf_{x \in \Sigma} \frac{1}{|\nabla H(x)|} = \frac{2b}{aM}.
 \end{aligned}$$

Then (1.12) holds.  $\square$

The following lower bound estimate for the periods of periodic solutions of (1.1) on  $\Sigma$  was proved in [4]. Note that the conditions  $(V_1)$  and  $(V_2)$  are not necessary in this result.

**Lemma 1.5.** (See [1,4].) Suppose  $\Sigma$  is a  $C^1$  positive-type hypersurface transverse to a linear positive vector field  $V$  satisfying (1.3),  $H$  is defined by (1.6). Then any nontrivial  $T$ -periodic solution of (1.1) on  $\Sigma$  satisfies  $T \geq \pi r_0^2 a^2 / b^2$ .

The main result of this paper is as follows

**Theorem 1.6.** Suppose  $\Sigma$  is a  $C^2$  positive-type hypersurface transverse to a vector field  $V$  defined by (1.2), then  $\Sigma$  carries at least  $n$  periodic solutions of (1.1) provided one of the following conditions holds:

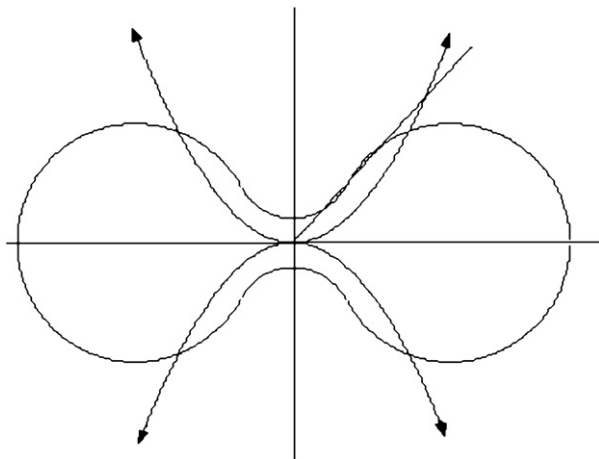
- (i)  $\frac{a+b}{a} \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}} < 2r_0^2 a^2 / b^2$ .
- (ii)  $R^2 < 2 \left(\frac{\sqrt{6}a^2 r_0}{(a+b)b}\right)^{\frac{2a}{a+b}} \left(\frac{r^2}{2}\right)^{\frac{a}{2b}}$ .

**Remark.** If  $\Sigma$  is convex or star-shaped, set  $V(x) = x$ . The conditions (i) and (ii) of Theorem 1.6 become  $R < \sqrt{2}r_0$  and  $R^2 < \sqrt{3}rr_0$  respectively. Therefore, Theorem 1.6 concludes the main result of [9] (Theorem 2 of Section IV) and Theorem 3 of [12]. The following examples show that our result is more general essentially.

**Example 1.7.** We construct an example of hypersurface on  $\mathbf{R}^2$ . Denote the coordinate of  $\mathbf{R}^2$  by  $(p, q)$ . Define four smooth arcs  $\sigma_i$ ,  $i = 1, 2, 3, 4$ , as follows:

$$\begin{aligned}
 \sigma_1 &= \{(p, q) \mid (p - \sqrt{7})^2 + q^2 = 4, p \in [\sqrt{7}/3, \sqrt{7} + 2]\}, \\
 \sigma_2 &= \{(p, q) \mid q = \sqrt{2} - \sqrt{1 - p^2}, p \in [-\sqrt{7}/3, \sqrt{7}/3]\}, \\
 \sigma_3 &= \{(p, q) \mid (p + \sqrt{7})^2 + q^2 = 4, p \in [-\sqrt{7} + 2, -\sqrt{7}/3]\}, \\
 \sigma_4 &= \{(p, q) \mid q = -\sqrt{2} + \sqrt{1 - p^2}, p \in [-\sqrt{7}/3, \sqrt{7}/3]\}.
 \end{aligned}$$

Then  $\{\sigma_i\}_{i=1}^4$  form a  $C^1$  closed curve  $\Sigma$  on  $\mathbf{R}^2$ . Define the positive vector field  $V$  on  $\mathbf{R}^2$  by  $V(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x$ . It can be proved that  $V$  is transverse to  $\Sigma$  and  $\Sigma$  is not star-shaped with respect to any point. Moreover,  $\Sigma$  is not a normalized positive-type hypersurface defined in [2]. We omit the elementary and trivial details. In Fig. 1 we draw the graph of  $\Sigma$ , four orbits of  $V$  and a ray starting from origin, crossing the connecting point  $(\sqrt{7}/3, 2\sqrt{2}/3)$  of  $\sigma_1$  and  $\sigma_2$ , and intersecting  $\Sigma$  three times.

Fig. 1. Graph of  $\Sigma$  of Example 1.7.

**Example 1.8.** (An example of positive-type hypersurface in  $\mathbf{R}^4$ .) Suppose the parametric equation of the planar closed curve defined in Example 1.7 is

$$\begin{cases} p = \rho(\theta), \\ q = \sigma(\theta), \end{cases} \quad \theta \in [0, 2\pi]. \quad (1.13)$$

Denote the coordinate of  $\mathbf{R}^4$  by  $(p_1, p_2, q_1, q_2)$ . Define a hypersurface  $\Sigma$  in  $\mathbf{R}^4$  as follows:

$$\Sigma: \begin{cases} p_1 = \sqrt{1 - (r^2 + s^2)} \rho(\theta), \\ p_2 = r, \\ q_1 = \sqrt{1 - (r^2 + s^2)} \sigma(\theta), \\ q_2 = s, \end{cases} \quad \begin{matrix} \theta \in [0, 2\pi], \\ r^2 + s^2 \leq 1. \end{matrix} \quad (1.14)$$

It is easy to see that the intersection of  $\Sigma$  and the  $(p_1, q_1)$ -subspace  $\{x = (p_1, 0, q_1, 0) \mid p_1, q_1 \in \mathbf{R}^1\}$  is just the planar closed curve of Example 1.7, the intersection of  $\Sigma$  and the other  $(p_i, q_j)$ -subspaces are ellipses (or circles).  $\Sigma$  is symmetric with respect to  $(p_1, q_1)$ -subspace. By using the method in Example 3.1 of [3], it can be proved that  $\Sigma$  is not star-shaped with respect to any point in  $\mathbf{R}^4$ . Define a vector  $V$  on  $\mathbf{R}^4$  with form (1.2) by

$$V(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} x, \quad x \in \mathbf{R}^4. \quad (1.15)$$

Then  $V$  is transverse to  $\Sigma$ , which implies that  $\Sigma$  satisfies the condition of Theorem 1.6.

## 2. Proof of Theorem 1.6

We use critical point theory to prove Theorem 1.6. As in [1,6], identify  $\mathbf{R}^{2n}$  with  $\mathbf{C}^n$  through the isomorphism  $(p, q) \rightarrow p + iq$ .  $\mathbf{C}^n$  has the Hermitian inner product  $\langle \xi, \eta \rangle_{\mathbf{C}^n} = \sum_{j=1}^n \xi_j \bar{\eta}_j$

with corresponding norm  $\|\cdot\|$ . Let  $\{e_1, e_2, \dots, e_{2n}\}$  be the standard basis of  $\mathbf{R}^{2n}$  and  $\phi_j = e_j + ie_{n+j}$ ,  $j = 1, \dots, n$ . Then  $\{\phi_1, \dots, \phi_n\}$  is the standard basis of  $\mathbf{C}^n$ . A function  $G \in C^1(\mathbf{R}^{2n}, \mathbf{R}^1)$  induces a function  $G: \mathbf{C}^n \rightarrow \mathbf{R}^1$  defined by  $G(u) = G(p, q)$ , where  $u = p + iq$ . Denote

$$\begin{aligned}\nabla G(p, q) &= (G_p(p, q), G_q(p, q)), \\ \nabla G(u) &= G_p(p, q) + iG_q(p, q).\end{aligned}$$

The Hamiltonian system

$$\dot{x} = J\nabla H(x), \quad x = (p, q),$$

can be rewritten in complex form as follows

$$-i\dot{z} = \nabla H(z), \quad z = p + iq. \quad (2.1)$$

Denote  $\mathbf{S}^1 = \mathbf{R}^1/2\pi\mathbf{Z}$ . Let  $\mathbf{L} = \mathbf{L}^2(\mathbf{S}^1, \mathbf{C}^n)$  with the usual inner product

$$\langle z(t), w(t) \rangle_{\mathbf{L}} = \frac{1}{2\pi} \int_0^{2\pi} \langle z(t), \bar{w}(t) \rangle_{\mathbf{C}^n} dt$$

and the norm  $\|z\|_{\mathbf{L}}$ . Every  $z \in \mathbf{L}$  has Fourier expansion  $z(t) = \sum_{k \in \mathbf{Z}} e^{ikt} z_k$ , where  $z_k = \sum_{j=1}^n \langle u_{kj}, z \rangle_{\mathbf{L}} \phi_j \in \mathbf{C}^n$  and

$$\{u_{kj}(t) = e^{ikt} \phi_j \mid k \in \mathbf{Z}, j = 1, \dots, n\}.$$

Let  $\mathbf{E} = \mathbf{W}^{1/2,2}(\mathbf{S}^1, \mathbf{C}^n)$  be the Sobolev space with the inner product

$$\langle z, w \rangle_{\mathbf{E}} = \sum_{k \in \mathbf{Z}} (1 + |k|) \langle z_k, w_k \rangle_{\mathbf{C}^n}$$

and corresponding norm  $\|z\|_{\mathbf{E}} = \langle z, z \rangle_{\mathbf{E}}^{1/2}$ . Let

$$\mathbf{E}_k = \text{span}\{e^{ikt} \phi_j \mid j = 1, \dots, n\}$$

for every  $k \in \mathbf{Z}$ ,  $\mathbf{E}^+ = \bigoplus_{k>0} \mathbf{E}_k$ ,  $\mathbf{E}^- = \bigoplus_{k<0} \mathbf{E}_k$  and  $\mathbf{E}^0 = \mathbf{E}_0$ . Then  $\mathbf{E} = \mathbf{E}^+ \oplus \mathbf{E}^0 \oplus \mathbf{E}^-$ . Denote by  $P^+$ ,  $P^0$  and  $P^-$  the orthogonal projections on  $\mathbf{E}^+$ ,  $\mathbf{E}^0$  and  $\mathbf{E}^-$ , respectively.

Define the operator  $A = -i \frac{d}{dt} : \text{dom}(A) \subset \mathbf{L} \rightarrow \mathbf{L}$  with  $\text{dom}(A) = \mathbf{W} = \mathbf{W}^{1,2}(\mathbf{S}^1, \mathbf{C}^n)$ . It is well known that  $\sigma(A) = \mathbf{Z}$ , that  $\mathbf{E}_k$  is the eigenspace of  $A$  with eigenvalue  $k$  and that  $\ker A = \mathbf{C}^n$ . The self-adjoint extension  $\tilde{A}$  of  $A$  on  $\mathbf{E}$  is defined by

$$\langle \tilde{A}z, w \rangle_{\mathbf{E}} = \sum_{k \in \mathbf{Z}} k \langle z_k, w_k \rangle_{\mathbf{C}^n}, \quad \forall z, w \in \mathbf{E}. \quad (2.2)$$

Then  $\tilde{A}$  is a bounded linear operator and  $\sigma(\tilde{A}) = \{k/(1 + |k|) \mid k \in \mathbf{Z}\}$ .

Define a functional  $I$  on  $\mathbf{E}$  by

$$I(z) = \frac{1}{2} \langle \tilde{A}z, z \rangle_{\mathbf{E}}. \quad (2.3)$$

Let  $\mathbf{L}_r = (\mathbf{L}, \langle \cdot, \cdot \rangle_{\mathbf{L}_r})$  and  $\mathbf{E}_r = (\mathbf{E}, \langle \cdot, \cdot \rangle_{\mathbf{E}_r})$ , with inner products  $\langle \cdot, \cdot \rangle_{\mathbf{L}_r} = \operatorname{Re} \langle \cdot, \cdot \rangle_{\mathbf{L}}$  and  $\langle \cdot, \cdot \rangle_{\mathbf{E}_r} = \operatorname{Re} \langle \cdot, \cdot \rangle_{\mathbf{E}}$ , respectively.

Define  $f: \mathbf{L}_r \rightarrow \mathbf{R}^1$  by

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} H(z(t)) dt. \quad (2.4)$$

By Lemma 1.4,  $f \in C^{1, \text{lip}}(\mathbf{L}_r, \mathbf{R}^1)$  and

$$\langle \nabla f(z), w \rangle_{\mathbf{L}_r} = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \langle \nabla H(z), w \rangle_{\mathbf{C}^n} dt.$$

Define the operator  $F$  on  $\mathbf{E}$  by

$$\langle F(z), w \rangle_{\mathbf{E}_r} = \langle \nabla f(z), w \rangle_{\mathbf{L}_r}. \quad (2.5)$$

Let  $V$  be the linear positive vector field of (1.2). The complexification  $\hat{V}$  of  $V$  is

$$\hat{V}(p + iq) = ap + ibq. \quad (2.6)$$

Define an isomorphism  $\tilde{V}$  on  $\mathbf{E}$  by  $(\tilde{V}z)(t) = \hat{V}z(t)$ . Denoted by  $\{\tilde{\varphi}_s\}$  the flow of  $\tilde{V}$ , then

$$\tilde{\varphi}_s(p + iq) = e^{as}p + e^{bs}iq. \quad (2.7)$$

**Lemma 2.1.** *For every  $z \in \mathbf{E}_r$ , there hold*

$$\langle \tilde{A}\tilde{V}z, \tilde{V}z \rangle_{\mathbf{E}_r} = ab \langle \tilde{A}z, z \rangle_{\mathbf{E}_r}, \quad (2.8)$$

$$\langle \tilde{A}z, \tilde{V}z \rangle_{\mathbf{E}_r} = \frac{a+b}{2} \langle \tilde{A}z, z \rangle_{\mathbf{E}_r}, \quad (2.9)$$

$$\langle \tilde{A}\tilde{\varphi}_s z, \tilde{\varphi}_s z \rangle_{\mathbf{E}_r} = e^{(a+b)s} \langle \tilde{A}z, z \rangle_{\mathbf{E}_r}, \quad (2.10)$$

$$2af(z) \leq \langle F(z), \tilde{V}z \rangle_{\mathbf{E}_r} \leq 2bf(z). \quad (2.11)$$

**Proof.** Let  $z = p + iq \in \operatorname{dom}(A) = \mathbf{W}$ . Then

$$\tilde{V}z = ap + ibq, \quad \tilde{A}\tilde{V}z = b\dot{q} - ia\dot{p}.$$

Hence,



$$\begin{aligned}
\langle \tilde{A}\tilde{V}z, \tilde{V}z \rangle_{\mathbf{E}_r} &= \operatorname{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \langle b\dot{q}(t) - ia\dot{p}(t), \overline{ap(t) + ibq(t)} \rangle_{\mathbf{C}^n} dt \right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} (ab\langle p(t), \dot{q}(t) \rangle_{\mathbf{R}^n} - ab\langle \dot{p}(t), q(t) \rangle_{\mathbf{R}^n}) dt \\
&= ab\langle \tilde{A}z, z \rangle_{\mathbf{E}_r}.
\end{aligned}$$

By the fact that  $\mathbf{W}$  is dense in  $\mathbf{E}$ , (2.8) holds. All of other equalities and inequalities come from the direct computations, we omit the details.  $\square$

By (2.8) we have the following important corollary:

**Corollary 2.2.**  $\mathbf{E}^+$ ,  $\mathbf{E}^-$  and  $\mathbf{E}^0$  are invariant subspaces of  $\tilde{V}$ .

Let  $\tilde{\Sigma} = \{z \in \mathbf{E}_r \mid f(z) = 1\}$ . We use the relative index theory introduced in [6] to find the critical points of  $I|_{\tilde{\Sigma}}$  with positive critical values. The complex Hilbert space  $\mathbf{E}$  has a natural  $\mathbf{S}^1$ -action  $\mathcal{T} = \{\mathcal{T}_\theta \mid \theta \in \mathbf{S}^1\}$  defined by  $\mathcal{T}_\theta z(t) = z(t + \theta)$ . The space of fixed points of  $\mathcal{T}$  is  $\mathbf{E}^0$ .  $\mathbf{E}^+$  is a closed linear subspace invariant under  $\mathcal{T}$ ,  $\mathbf{E}^0 \subset \mathbf{X}^\perp$ . Under this  $\mathbf{S}^1$ -action,  $I$  and  $f$  are invariant functionals,  $\tilde{\Sigma}$  is an invariant set and the linear operators  $\tilde{A}$  and  $\tilde{V}$  are equivariant. Let

$$\begin{aligned}
\mathcal{G} &= \{B \subset \mathbf{E} \mid B \cap \mathbf{E}^0 = \emptyset, \text{ is closed and invariant under } \mathcal{T}\}, \\
\mathcal{F} &= \{B \subset \mathbf{E} \mid B \subset \mathbf{E} \setminus \{0\}, B \text{ is closed and invariant under } \mathcal{T}\}.
\end{aligned}$$

For  $B \in \mathcal{G}$  (respectively  $B \in \mathcal{F}$ ), denote by  $\gamma(B)$  and  $\gamma_r(B)$  the index and the relative index of  $B$  with respect to  $\mathbf{E}^+$  defined in Section 2 of [6] respectively.

**Lemma 2.3.** The functional  $I|_{\tilde{\Sigma}}$  satisfies the (PS) condition.

**Proof.** Suppose  $\{u_n\} \subset \tilde{\Sigma}$  is a sequence such that  $\{I(u_n)\}$  is bounded and  $\nabla I|_{\tilde{\Sigma}}(u_n) \rightarrow 0$ . Then there exist  $\lambda_n \in \mathbf{R}^1$  such that

$$z_n := \tilde{A}u_n - \lambda_n F(u_n) \rightarrow 0 \quad \text{in } \mathbf{E}_r. \quad (2.12)$$

Decompose  $u_n$  as

$$u_n = u_n^+ + u_n^- + u_n^0 \in \mathbf{E}^+ \oplus \mathbf{E}^- \oplus \mathbf{E}^0. \quad (2.13)$$

Since  $\{I(u_n)\}$  is a bounded sequence, there exist constants  $\alpha, \beta$  and  $d$  such that

$$-d + \alpha \|u_n^+\|_{\mathbf{E}_r} \leq \|u_n^-\|_{\mathbf{E}_r} \leq \beta \|u_n^+\|_{\mathbf{E}_r} + d. \quad (2.14)$$

Multiplying (2.12) by  $\tilde{V}u_n$ ,

$$\langle z_n, \tilde{V}u_n \rangle_{\mathbf{E}_r} = \langle \tilde{A}u_n, \tilde{V}u_n \rangle_{\mathbf{E}_r} - \lambda_n \langle F(u_n), \tilde{V}u_n \rangle_{\mathbf{E}_r}. \quad (2.15)$$

By (2.9) and (2.11),  $\{\langle \tilde{A}u_n, \tilde{V}u_n \rangle_{\mathbf{E}_r}\}$  is bounded,  $\langle F(u_n), \tilde{V}u_n \rangle_{\mathbf{E}_r} \geq 2a$ . Hence, there exist constants  $d_1$  and  $d_2$  such that

$$|\lambda_n| \leq \frac{1}{2a} (|\langle \tilde{A}u_n, \tilde{V}u_n \rangle_{\mathbf{E}_r}| + \|z_n\|_{\mathbf{E}_r} \cdot \|\tilde{V}u_n\|_{\mathbf{E}_r}) \leq d_1 + d_2 \|u_n\|_{\mathbf{E}_r}.$$

Note that  $\frac{1}{2}\|u\|_{\mathbf{E}_r}^2 \leq \langle \tilde{A}u, u \rangle_{\mathbf{E}_r}$  for every  $u \in \mathbf{E}^+$ ,  $\{\|u_n\|_{\mathbf{L}_r}\}$  is bounded, then one has

$$\begin{aligned} \|u_n^+\|_{\mathbf{E}_r}^2 &\leq 2\langle \tilde{A}u_n^+, u_n^+ \rangle_{\mathbf{E}_r} = 2\langle \tilde{A}u_n, u_n^+ \rangle_{\mathbf{E}_r} \\ &= 2\langle z_n + \lambda_n F(u_n), u_n^+ \rangle_{\mathbf{E}_r} \\ &\leq 2\|z_n\|_{\mathbf{E}_r} \cdot \|u_n\|_{\mathbf{E}_r} + 2|\lambda_n| \cdot |\langle \nabla f(u_n), u_n^+ \rangle_{\mathbf{L}_r}| \\ &\leq 2\|z_n\|_{\mathbf{E}_r} \cdot \|u_n\|_{\mathbf{E}_r} + 2|\lambda_n| \cdot \|\nabla f(u_n)\|_{\mathbf{L}_r} \|u_n\|_{\mathbf{L}_r} \\ &\leq d_3 + d_4 \|u_n\|_{\mathbf{E}_r}, \end{aligned}$$

where  $d_i, i = 3, 4$ , are constants. In view of (2.12),  $\|u_n\|_{\mathbf{E}_r}$  is bounded and so is  $\{\lambda_n\}$ . Since  $\tilde{A}$  is a compact perturbation of  $P^+ - P^-$  and  $F$  is compact, by standard method,  $\{u_n\}$  has a convergent subsequence in  $\mathbf{E}$ .  $\square$

Define  $\Lambda = \{z \in \mathbf{E} \mid \frac{1}{2}\|z\|_{\mathbf{L}}^2 = 1\}$ . Repeat the proof of Lemma 3.6 of [1], it is easy to see that for every  $z \in \tilde{\Sigma}$ , there exists a unique  $\psi(z) \in \mathbf{R}^1$  such that  $\tilde{\varphi}_{\psi(z)}z \in \Lambda$ , and  $\psi(z)$  is  $C^1$  invariant function. Define  $\tilde{P}: \tilde{\Sigma} \rightarrow \Lambda$  along  $\tilde{V}$  by

$$\tilde{P}(z) = \tilde{\varphi}_{\psi(z)}z. \quad (2.16)$$

Then  $\tilde{P}$  is an equivariant  $C^1$  diffeomorphism.

For any  $z \in \tilde{\Sigma}$ ,

$$\frac{1}{2}\|\tilde{\varphi}_{\psi(z)}z\|_{\mathbf{L}_r}^2 = 1 = f(z).$$

By (1.4), (1.11) and (2.4), for any  $z \in \mathbf{E}$ ,

$$\begin{aligned} e^{2a\psi(z)}\|z\|_{\mathbf{L}_r}^2 &\leq \|\tilde{\varphi}_{\psi(z)}z\|_{\mathbf{L}_r}^2 \leq e^{2b\psi(z)}\|z\|_{\mathbf{L}_r}^2, \\ \|z\|_{\mathbf{L}_r}^2/R^2 &\leq f(z) \leq \|z\|_{\mathbf{L}_r}^2/r^2. \end{aligned}$$

Hence,

$$e^{2b\psi(z)} \geq 2/R^2, \quad e^{2a\psi(z)} \leq 2/r^2, \quad \forall z \in \tilde{\Sigma}.$$

Consequently,

$$(2/R^2)^{\frac{1}{2b}} \leq e^{\psi(z)} \leq (2/r^2)^{\frac{1}{2a}}. \quad (2.17)$$

**Lemma 2.4.** For any  $B \in \{D \in \mathcal{F} \mid D \subset \tilde{\Sigma}\}$ , there holds

$$\gamma_r(\tilde{P}(B)) = \gamma_r(B).$$

**Proof.** For every  $z \in \mathbf{E} \setminus \{0\}$ ,  $\tilde{\Sigma} \cap \{\tilde{\varphi}_s z\}_{s=-\infty}^{+\infty}$  has exactly one element, denote it by  $\sigma(z)$ . Extend  $\tilde{P}$  to  $\mathbf{E} \setminus \{0\}$  by

$$\tilde{P}(z) = \tilde{\varphi}_{\psi(\sigma(z))} z, \quad z \in \mathbf{E} \setminus \{0\}.$$

Then  $\tilde{P}: \mathbf{E} \setminus \{0\} \rightarrow \mathbf{E} \setminus \{0\}$  is invertible. By (2.17) and Proposition 3.2 of [1], the conclusion holds.  $\square$

Denote by  $\{\mu_k\}_{k \in \mathbf{Z}}$  the set of eigenvalues of  $A = -i \frac{d}{dt}$ , and by  $\{\vartheta_k\}_{k \in \mathbf{Z}}$  the set of corresponding eigenfunctions satisfying

$$\langle A \vartheta_j, \vartheta_k \rangle_{\mathbf{L}} = \delta_{jk}, \quad j, k \in \mathbf{Z},$$

where  $\delta_{jk}$  is equal to 1 if  $j = k$  and 0 otherwise. It is well known that

$$\mu_1 = \cdots = \mu_n = 1. \quad (2.18)$$

We give a minimax-type description for  $\{\mu_k\}_{k > 0}$  that will be used to estimate the critical values of  $I|_{\tilde{\Sigma}}$ . There are many ways to give such description, see for example [1,6].

**Lemma 2.5.** The positive eigenvalues  $\mu_k$  of problem  $Az = \mu z$  are given by

$$\mu_k = \inf_{B \in \Gamma_k(\Lambda)} \sup_B I(z), \quad (2.19)$$

where  $\Gamma_k(\Lambda) = \{B \in \mathcal{F} \mid B \subset \Lambda, \gamma_r(B) \geq k\}$ .

**Proof.** The minimax-type description (2.19) and the descriptions given in [1,6] are equivalent essentially. We give the proof briefly for completeness. Note that  $z = \sum_{j \in \mathbf{Z}} \xi_j \vartheta_j \in \Lambda$  if and only if  $\frac{1}{2} \sum_{j \in \mathbf{Z}} \xi_j^2 = 1$ . Let

$$B_k = \Lambda \cap \text{span}\{\vartheta_j \mid j \leq k\}.$$

Then  $\gamma_r(B_k) = k$  and

$$\frac{1}{2} \langle Az, z \rangle_{\mathbf{L}} = \frac{1}{2} \sum_{j \leq k} \xi_j^2 \mu_j \leq \mu_k, \quad z \in B_k.$$

Hence,

$$\inf_{B \in \Gamma_k(\Lambda)} \sup_B I(z) \leq \sup_{B_k} I(z) \leq \mu_k.$$

On the other hand, if  $B \in \Gamma_k(\Lambda)$ , by Corollary 2.9 of [6],

$$B \cap \text{span}\{\vartheta_j \mid j \geq k\} \neq \emptyset.$$

Let  $v = \sum_{j \geq k} v_j \vartheta_j \in B \cap \text{span}\{\vartheta_j \mid j \geq k\}$ . Then

$$\sup_B I(z) \geq I(v) = \frac{1}{2} \sum_{j \geq k} |v_j|^2 \mu_j \geq \mu_k,$$

which implies

$$\inf_{B \in \Gamma_k(\Lambda)} \sup_B I(z) \geq \mu_k.$$

Therefore, (2.19) holds.  $\square$

We construct the critical values of  $I|_{\tilde{\Sigma}}$ . Denote

$$\begin{aligned} \Gamma_k(\tilde{\Sigma}) &= \{B \in \mathcal{F} \mid B \subset \tilde{\Sigma}, \gamma_r(B) \geq k\}, \\ K_c &= \{z \in \tilde{\Sigma} \mid \nabla I|_{\tilde{\Sigma}}(z) = 0, I(z) = c\}. \end{aligned}$$

**Lemma 2.6.** *Let*

$$c_k = \inf_{B \in \Gamma_k(\tilde{\Sigma})} \sup_{z \in B} I(z), \quad k \in \mathbf{N}. \quad (2.20)$$

*Then  $K_{c_k} \neq \emptyset$ . Moreover, if there exist  $k, p \in \mathbf{N}$  such that*

$$c_k = c_{k+1} = \cdots = c_{k+p-1} := c,$$

*then  $\gamma(K_c) \geq p$ .*

**Proof.** *Step 1.* We construct a mapping  $\eta: \mathbf{E} \rightarrow \mathbf{E}$  with the following properties:

- (i)  $\eta(z) = Lz + Kz$ , where  $K$  is compact,  $L$  is an equivariant isomorphism and  $L((\mathbf{E}^+)^\perp) = (\mathbf{E}^+)^\perp$ .
- (ii)  $\eta: \mathbf{E} \rightarrow \mathbf{E}$  is an equivariant homeomorphism.
- (iii) There exists a  $\delta > 0$ , such that

$$\eta(\{I \leq c + \epsilon\} \setminus N_\delta(K_c)) \subset \{I \leq c - \epsilon\},$$

where  $N_\delta(K_c) = \{z \in \tilde{\Sigma} \mid \text{dist}(z, K_c) \leq \delta\}$ ,  $\{I \leq a\} = \{z \in \tilde{\Sigma} \mid I(z) \leq a\}$ .

- (iv) For  $B \in \Gamma_k(\tilde{\Sigma})$ ,  $\eta(B) \in \Gamma_k(\tilde{\Sigma})$ .

Consider the initial-value problem

$$\begin{cases} \frac{d\xi}{dt} = -\tilde{A}\xi(t) + \frac{(\tilde{A}\xi(t), F(\xi(t)))_{\mathbf{E}_r}}{\|F(\xi(t))\|_{\mathbf{E}}^2} F(\xi(t)), \\ \xi(0) = u \in \mathbf{E} \setminus \{0\} \end{cases}$$

on  $\mathbf{E}$ . The flow  $\xi(t, z)$  of this problem is globally defined and leaves invariant the set  $\tilde{\Sigma}$ . By Lemma 3.3 of [5],  $\xi(t, z)$  is of the form

$$\xi(t, z) = e^{-t\tilde{A}}z + K(t, z)$$

with  $K(t, \cdot)$  compact. By Theorem 3.4 of [5], there exists a  $t_0 > 0$  such that

$$\eta(z) = \xi(t_0, z) = e^{-t_0\tilde{A}}z + K(t_0, z) = Lz + Kz,$$

with  $L = e^{-t_0\tilde{A}}$ ,  $K = K(t_0, \cdot)$ , satisfies (i)–(iii).

By Propositions 2.11, 2.12 of [6], for  $B \in \Gamma_k(\tilde{\Sigma})$ ,

$$\begin{aligned} \gamma_r(\eta(B)) &= \gamma_r((L + K)(B)) = \gamma_r(L(I + L^{-1}K)(B)) \\ &\geq \gamma_r((I + L^{-1}K)(B)) \geq \gamma_r(B). \end{aligned}$$

Then (iv) holds.

*Step 2.* By Lemma 2.3,  $K_c$  is compact. Suppose  $\gamma(K_c) \leq p - 1$ . By the definition of  $c$ , there exists  $\mathcal{A} \in \Gamma_{k+p-1}(\tilde{\Sigma})$  such that

$$\sup_{\mathcal{A}} I(z) \leq c + \epsilon.$$

Let  $B = \eta(\overline{\mathcal{A} \setminus N_\delta(K_c)})$  and let  $N_\delta(K_c)$  satisfy  $\gamma(N_\delta(K_c)) = \gamma(K_c)$ . By (iv) of step 1 and Corollary 2.15 of [6], we have

$$\begin{aligned} \gamma_r(B) &= \gamma_r(\eta(\overline{\mathcal{A} \setminus N_\delta(K_c)})) \geq \gamma_r(\overline{\mathcal{A} \setminus N_\delta(K_c)}) \\ &\geq \gamma_r(\mathcal{A}) - \gamma(N_\delta(K_c)) \geq k. \end{aligned}$$

Hence,  $B \in \Gamma_k(\tilde{\Sigma})$  and consequently,  $\sup_B I(z) \geq c$ . By property (iii) of  $\eta$ ,  $B \subset \{I \leq c - \epsilon\}$ , which is a contradiction.  $\square$

**Lemma 2.7.** *The critical values defined by (2.20) satisfy*

$$0 < \left(\frac{r^2}{2}\right)^{\frac{a+b}{2b}} \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}}. \quad (2.21)$$

**Proof.** For any  $z \in \tilde{\Sigma}$ ,  $\tilde{P}(z) = \tilde{\varphi}_{\psi(z)}z \in \Lambda$ . By (2.10),

$$\begin{aligned} I(\tilde{P}(z)) &= \frac{1}{2} \langle \tilde{A}(\tilde{\varphi}_{\psi(z)} z), \tilde{\varphi}_{\psi(z)} z \rangle_{\mathbf{E}} \\ &= \frac{1}{2} e^{(a+b)\psi(z)} \langle \tilde{A}z, z \rangle_{\mathbf{E}} = e^{(a+b)\psi(z)} I(z). \end{aligned}$$

Then by (2.17),

$$\left(\frac{r^2}{2}\right)^{\frac{a+b}{2b}} I(\tilde{P}(z)) \leq I(z) \leq \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}} I(\tilde{P}(z)).$$

Thus, by Lemma 2.4, (2.19) and (2.20),

$$\left(\frac{r^2}{2}\right)^{\frac{a+b}{2b}} \mu_k \leq c_k \leq \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}} \mu_k.$$

Since  $\mu_k = 1$  for  $k = 1, \dots, n$ , (2.21) holds.  $\square$

**Lemma 2.8.** Suppose  $z \in \tilde{\Sigma}$  is a critical point of  $I|_{\tilde{\Sigma}}$  with critical value  $c = I(z) > 0$ ,  $\lambda$  is the Lagrange multiplier of  $z$ . Then  $u(t) = z(\frac{t}{\lambda})$  is a periodic solution of (1.1) on  $\Sigma$  with period in  $[\frac{a+b}{b}\pi c, \frac{a+b}{a}\pi c]$ .

**Proof.** Since  $\tilde{A}z = \lambda Fz$ , it is easy to see that  $u(t) = z(\frac{t}{\lambda})$  satisfies (2.1). Denote  $u = p + iq$ . The corresponding real form  $u = (p, q)$  is a  $2\pi\lambda$ -periodic solution of (1.1) on  $\Sigma$ .

By (2.9) and (2.11),

$$\begin{aligned} \langle \tilde{A}z, \tilde{V}z \rangle_{\mathbf{E}_r} &= \frac{a+b}{2} \langle \tilde{A}z, z \rangle_{\mathbf{E}_r} = (a+b)c, \\ 2a &\leq \langle F(z), \tilde{V}z \rangle_{\mathbf{E}_r} \leq 2b. \end{aligned}$$

Hence, by the equality

$$\langle \tilde{A}z, \tilde{V}z \rangle_{\mathbf{E}_r} = \lambda \langle F(z), \tilde{V}z \rangle_{\mathbf{E}_r},$$

one has

$$\frac{a+b}{2b} I(z) \leq \lambda \leq \frac{a+b}{2a} I(z). \quad (2.22)$$

Then the conclusion holds.  $\square$

Now we give the proof of Theorem 1.6.

**Proof of Theorem 1.6.**  $I|_{\tilde{\Sigma}}$  has  $n$  positive critical values  $c_1, \dots, c_n$  satisfying (2.21). No loss of generality, we suppose they are distinct mutually. Otherwise, if two (or more) of them coincide, then the corresponding critical set has index greater than or equal to 2, so that (1.1) has infinitely many periodic solutions on  $\Sigma$ .

Case 1.  $\frac{a+b}{a} \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}} < 2r_0^2 a^2 / b^2$ .

Denote by  $z_1, z_2, \dots, z_n$  the critical points corresponding to the critical values  $c_1, c_2, \dots, c_n$ , by  $\lambda_1, \lambda_2, \dots, \lambda_n$  the corresponding Lagrange multipliers. Let  $u_k(t) = z_k(t/\lambda_k)$ ,  $i = 1, \dots, n$ . By Lemma 2.8,  $u_k$  is a periodic solution of (1.1) on  $\Sigma$  with periods  $T_k \in [\frac{a+b}{b}\pi c_k, \frac{a+b}{a}\pi c_k]$  for every  $k$ . Thus,

$$T_k \leq \frac{a+b}{a} \pi \left( \frac{R^2}{2} \right)^{\frac{a+b}{2a}} < 2\pi r_0^2 a^2 / b^2, \quad k = 1, \dots, n. \quad (2.23)$$

The inequality (2.23) and Lemma 1.5 imply that  $T_k$  is the minimal period of  $u_k$ . Hence  $u_k$  and  $u_j$  are geometrically distinct if  $k \neq j$ .

Case 2.  $R^2 < 2(\frac{\sqrt{6}a^2r_0}{(a+b)b})^{\frac{2a}{a+b}} (\frac{r^2}{2})^{\frac{a}{2b}}$ .

Use the same notations as above. Let  $z_k^*$  be the primitive of  $z_k$  defined by  $z_k^*(t) = z_k(t/m_k)$ , where  $m_k$  is the positive integer such that  $2\pi/m_k$  is the minimal period of  $z_k$ . It suffices to prove that  $z_k^* \neq z_j^*$  for  $k \neq j$ . At first we prove  $m_k \leq 2$  for  $k = 1, \dots, n$ .

Define  $\bar{z}_k = \frac{1}{2\pi} \int_0^{2\pi} z_k(t) dt$ ,  $\tilde{z}_k = z_k - \bar{z}_k$ . By Wirtinger inequality of [12],

$$\|\tilde{z}_k\|_{\mathbf{L}} \leq \frac{1}{m_k} \|\dot{\tilde{z}}_k\|_{\mathbf{L}}. \quad (2.24)$$

The Lagrange multiplier  $\lambda_k$  of  $z_k$  satisfies  $i\dot{z}_k = \lambda_k \nabla H(z_k)$ . Then by (2.21),

$$\begin{aligned} \left( \frac{r^2}{2} \right)^{\frac{a+b}{2b}} &\leq I(z_k) = \frac{1}{2} \langle \tilde{A}z_k, z_k \rangle_{\mathbf{E}} = \frac{1}{2} \langle Az_k, z_k \rangle_{\mathbf{L}} \\ &= \frac{1}{4\pi} \int_0^{2\pi} \langle i\dot{z}_k(t), z_k(t) \rangle_{\mathbf{C}^n} dt \\ &= \frac{1}{4\pi} \int_0^{2\pi} \langle i\dot{\tilde{z}}_k(t), \tilde{z}_k(t) \rangle_{\mathbf{C}^n} dt \\ &\leq \frac{1}{2} \|\tilde{z}_k\|_{\mathbf{L}} \|\dot{\tilde{z}}_k\|_{\mathbf{L}} \leq \frac{1}{2m_k} \|\dot{z}_k\|_{\mathbf{L}}^2 \\ &= \frac{1}{2m_k} \frac{1}{2\pi} \int_0^{2\pi} \|\lambda_k \nabla H(z_k)\|^2 \leq \frac{1}{2m_k} \lambda_k^2 M^2. \end{aligned}$$

By (1.12),  $\frac{1}{M} \geq \frac{ar_0}{2b}$ , one has

$$\lambda_k^2 \geq \frac{a^2 r_0^2 m_k}{2b^2} \left( \frac{r^2}{2} \right)^{\frac{a+b}{2b}}, \quad k = 1, \dots, n. \quad (2.25)$$

If  $m_k > 2$ , then (2.22) and (2.25) imply

$$\begin{aligned}
 I(z_k) &\geq \frac{2a}{a+b} \lambda_k \geq \frac{\sqrt{2m_k} a^2 r_0}{(a+b)b} \left(\frac{r^2}{2}\right)^{\frac{a+b}{4b}} \\
 &\geq \frac{\sqrt{6} a^2 r_0}{(a+b)b} \left(\frac{r^2}{2}\right)^{\frac{a+b}{4b}} > \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}},
 \end{aligned}$$

which contradicts (2.21).

Suppose  $z_k$  and  $z_j$  have the same primitive  $z_k^*$  for some  $k, j$ , then  $z_k^*(t) = z_k(t/m_k) = z_j(t/m_j)$ . Since  $m_k \leq 2, m_j \leq 2$  and  $m_k \neq m_j$ , there at most one of them, say  $m_k$ , is 2. Hence, by (2.25) with  $m_k = 2$ ,

$$\begin{aligned}
 I(z_j) &= 2I(z_k) \geq \frac{4a}{a+b} \lambda_k \\
 &\geq \frac{4a^2 r_0}{(a+b)b} \left(\frac{r^2}{2}\right)^{\frac{a+b}{4b}} > \left(\frac{R^2}{2}\right)^{\frac{a+b}{2a}}.
 \end{aligned}$$

It contradicts to (2.21) again.  $\square$

## Acknowledgments

The author expresses his sincere thanks to Dr. Haishen Lü for helping me draw Fig. 1 of this paper, to the referee for valuable suggestions and comments.

## References

- [1] T. An, Existence of multiple periodic orbits of Hamiltonian systems on positive-type hypersurfaces in  $\mathbf{R}^{2n}$ , J. Math. Anal. Appl. 278 (2003) 376–396.
- [2] T. An, On the number of periodic orbits of Hamiltonian systems on positive-type hypersurfaces in  $\mathbf{R}^{2n}$ , Nonlinear Anal. 56 (2004) 633–641.
- [3] T. An, The brake orbits of Hamiltonian systems on positive-type hypersurfaces, Positivity 10 (2006) 681–692.
- [4] T. An, G. Ye, The closed curves on positive-type hypersurfaces and the periods of nonlinear oscillations, J. Math. (PRC) 17 (1997) 527–532.
- [5] V. Benci, On the critical point theory for indefinite functions in the presence of symmetries, Trans. Amer. Math. Soc. 274 (1982) 533–572.
- [6] H. Berestycki, J.-M. Lasry, G. Mancini, B. Ruf, Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces, Comm. Pure Appl. Math. 38 (1985) 253–289.
- [7] Y. Dong, Y. Long, Closed characteristics on partial symmetric compact convex hypersurfaces in  $\mathbf{R}^{2n}$ , J. Differential Equations 196 (2004) 226–248.
- [8] I. Ekeland, H. Hofer, Convex Hamiltonian energy surfaces and their periodic trajectories, Comm. Math. Phys. 113 (1987) 419–469.
- [9] I. Ekeland, J.-M. Lasry, On the number of closed trajectories for a Hamiltonian flow on a convex energy surface, Ann. of Math. 112 (1980) 283–319.
- [10] P.L. Felmer, Periodic solutions of “superquadratic” Hamiltonian systems, J. Differential Equations 102 (1993) 188–207.
- [11] V.L. Ginzburg, An embedding  $S^{2n-1} \rightarrow \mathbf{R}^{2n}$ ,  $2n-1 > 7$ , whose Hamiltonian flow has no periodic trajectories, Int. Math. Res. Not. 2 (1995) 83–98.
- [12] M. Girardi, M. Matzeu, Solutions of minimal period for a class of nonconvex Hamiltonian systems and applications to the fixed energy problem, Nonlinear Anal. 10 (1986) 371–382.
- [13] H. Hofer, K. Wysocki, E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces, Ann. of Math. 148 (1998) 197–289.
- [14] H. Hofer, E. Zehnder, Periodic solutions on hypersurfaces and a result by C. Viterbo, Invent. Math. 90 (1987) 1–9.



- [15] X. Hu, Y. Long, Closed characteristics on non-degenerate star-shaped hypersurfaces in  $\mathbf{R}^{2n}$ , *Sci. China* 48 (8) (2002) 1038–1052.
- [16] C. Liu, Y. Long, C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in  $\mathbf{R}^{2n}$ , *Math. Ann.* 323 (2002) 201–215.
- [17] Y. Long, C. Zhu, Closed characteristics on compact convex hypersurfaces in  $\mathbf{R}^{2n}$ , *Ann. of Math.* 155 (2002) 317–368.
- [18] P.H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* 31 (1978) 157–184.
- [19] M. Struwe, Existence of periodic solutions of Hamiltonian systems on almost every energy surface, *Bol. Soc. Bras. Mat.* 20 (1990) 49–58.
- [20] C. Viterbo, A proof of Weinstein’s conjecture in  $\mathbf{R}^{2n}$ , *Ann. Inst. H. Poincaré Anal. Non Linéaire* 4 (1987) 337–356.
- [21] A. Weinstein, On the hypotheses of Rabinowitz’s periodic orbit theorems, *J. Differential Equations* 33 (1979) 353–358.
- [22] A. Weinstein, Periodic orbits for convex Hamiltonian systems, *Ann. of Math.* 108 (1978) 507–518.